SYNOPSIS OF Ph.D. THESIS TITLED

HIGHER ORDER GENERALIZED DERIVATIVES,APPROXIMATE DERIVATIVES AND THEIR PROPERTIES

SUBMITTED BY

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Synopsis:

The study of derivative was started many many years ago with the need of finding rate of change of a thing over the change of the parameter of dependence.There came 1st order derivative of a function of single variable.As time passed ,requirement as well as interest grew to study derivatives of multiple variables function ,higher order derivatives.Different type of functions' study were mixed up with derivatives to study easily the nature of a variety of functions and their characteristics like range,composition,symmetry, continuity,absolute continuity,uniform continuity,convexity,bounded variation,integrability of various type(Riemann,Laplace,Perron,Denjoy etc) and order ,different operations ,divided differences,sequence,series,semi-continuity etc.In this way the study of derivatives flew to numerous directions giving birth to a lot of new definitions of derivatives of a function ,involving the aforesaid connections of the function.These introduced a novel chapter of derivatives, titled as Generalized derivatives.These were interesting and claimed attention due to the conflict of property with normal derivatives ,that if f is a function such that f^n(x) exists then all lower order derivatives exist but it is not true for many of the newly evolved derivatives.Riemann derivatives or d.l.V.P. derivatives are examples of such derivatives.The generalized derivatives such as

- Peano-introduced by G.Peano and studied extensively by Genocchi, Poussin,Burkill,Denjoy,Zygmund,Oliver,Weil.
- Riemann,Riemann* By Denjoy, Corominas,Russel;used for studying higher order absolute continuity.
- d.l.V.P. introduced and studied by de la Vallee Poussin;to study Fourier series.
- Cesaro-originated from Cesaro summability of series,studied by Burkill,Sargent,Bergin;introduced CesaroPerron integral.

- Borel derivatives bybE.Borel;used to study trigonometricbseries.
- $L^p(\mathrm{A.\,P.\,Calderon, A.\,Zygmund})$
- Abel derivatives by Rajchman,Zygmund,Taylor;useful in Trigonometric series ,summability,integration etc.
- Laplace derivatives studied by him.
- Symmetric derivatives
- Approximate derivatives

And their interrelations and their properties as well as relation, etc got started being studied and explored a bunch of new ,useful and productive results.Many new results and characteristics of functions were found and proved .The process is going on.

We are interested in these generalized higher order derivatives,specially in Borel derivatives, L^p- derivatives , Abel derivatives and corresponding symmetric and approximate derivatives.The definition of these derivatives are as follows:

Borel Derivative:

Suppose f : R→ R be Lebesgue integrable in some neighbourhood of x ϵ R.
The right hand and left hand Borel derivatives of f at x is defined by

$$BD^+f(\mathrm{x})=\lim_{h\to 0+}\frac{1}{h}\int_0^h\frac{f(x+t)-f(x)}{t}dt$$

And

$$BD^-f(\mathrm{x})=\lim_{h\to 0+}\frac{1}{h}\int_{-h}^0\frac{f(x+t)-f(x)}{t}dt$$

The improper integral is assumed to be convergent. That is if the limit

$$\lim_{h\to 0+}\lim_{\epsilon\to 0+}\frac{1}{h}\int_\epsilon^h\frac{f(x+t)-f(x)}{t}dt$$

exists then the value of the limit is $BD^+f(\mathrm{x})$. Similarly

$$BD^-f(\mathrm{x})=\lim_{h\to 0+}\lim_{\epsilon\to 0+}\frac{1}{h}\int_{-h}^{-\epsilon}\frac{f(x+t)-f(x)}{t}dt$$

If $BD^+f(\mathrm{x})= BD^-f(\mathrm{x})$, then f is said to have Borel derivative at x and is denoted

by BDf(x). The right hand upper(lower) ordinary derivative of f is denoted by D^+f(x) ($\underline{D^+f}$(x)). The notion of other derivatives are similar. A function f is said to be Borel continuous at x if,

$$\lim_{h\to 0}\frac{1}{h}\int_0^h f(x+t)\,dt$$

Symmetric Borel Derivatives:

The first and second order symmetric Borel derivatives are respectively

$$SBD^1f(x)=\lim_{h\to 0}\frac{1}{h}\int_0^h \frac{f(x+t)-f(x-t)}{2t}\,dt$$

Similarly

$$SBD^2f(x)=\lim_{h\to 0}\frac{1}{h}\int_0^h \frac{f(x+t)+f(x-t)-2f(x)}{t^2}\,dt$$

Changing 'lim' into 'liminf' the de_nition of lower symmetric derivates and lower symmetric Borel derivates are obtained, similarly for the case of upper derivates.

L^p-Derivatives:

Let *f ϵ Lp* in some neighbourhood of a point x for some p, 1 ≤ p < ∞. If there is a polynomial *P(t) = P_x(t)* of degree at most k for which

$\frac{1}{h}\int_0^h |f(x+t)-P(x)|^p\,dt$ = o(h^{kp}) as h → 0

then f is said to have a kth *Lp*-derivative at x of order k and if a_k/k! is the coefficient of x^kin P(t) then a_k is called the kth *Lp*-derivative of f at x denoted by f_kp(x). If f_kp(x) exists then it is unique.

Abel Derivatives:
f (r , x)

$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\, r^n$

For a positive integer *k* ,

$\frac{\partial^k f(r,\,x)}{\partial x^k} = (-1)^{\frac{k}{2}} \sum_{n=1}^{\infty} (a_n\ n^k \cos nx + b_n\ n^k \sin nx)\, r^n$, *if k is even*

Or ,

$= (-1)^{\frac{(k+1)}{2}} \sum_{n=1}^{\infty} (a_n \ n^k \ \sin nx - b_n \ n^k \ cos \ nx) \ r^n$ *, if k is odd .*

$\limsup_{r \to 1-} \frac{\partial^k f(r, x)}{\partial x^k}$ is said to be the Upper Abel Derivative of f of order *k* at *x* and is denoted by $\overline{AD}_k \ f(x)$.

$\liminf_{r \to 1-} \frac{\partial^k f(r, x)}{\partial x^k}$ is said to be the Lower Abel Derivative of f of order *k* at *x* and is denoted by $\underline{AD}_k \ f(x)$.

If the values are equal , the common value is called the Abel Derivative of *f* of order *k* at *x* and is denoted by $AD_k \ f(x)$.

We shall study the derivatives farther and find out

- new relation between these derivatives ,
- which properties of ordinary derivative of a function are satisfied by these generalized higher order derivatives ,
- what could be extra properties for these new derivatives, if we can bring some change so that these derivatives satisfy general derivatives properties,
- how to explore and utilize various properties of a function like range,composition,symmetry,limit,continuity,absolute-continuity,uniform continuity,convexity,boundedvariation,integrability,differentiability of various type and order ,different operations ,divided differences,sequence,series,semi-continuity,BaireClass etc.
- Moreover we shall try to combine two or more generalized higher order derivatives to define new generalized higher order derivatives.

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